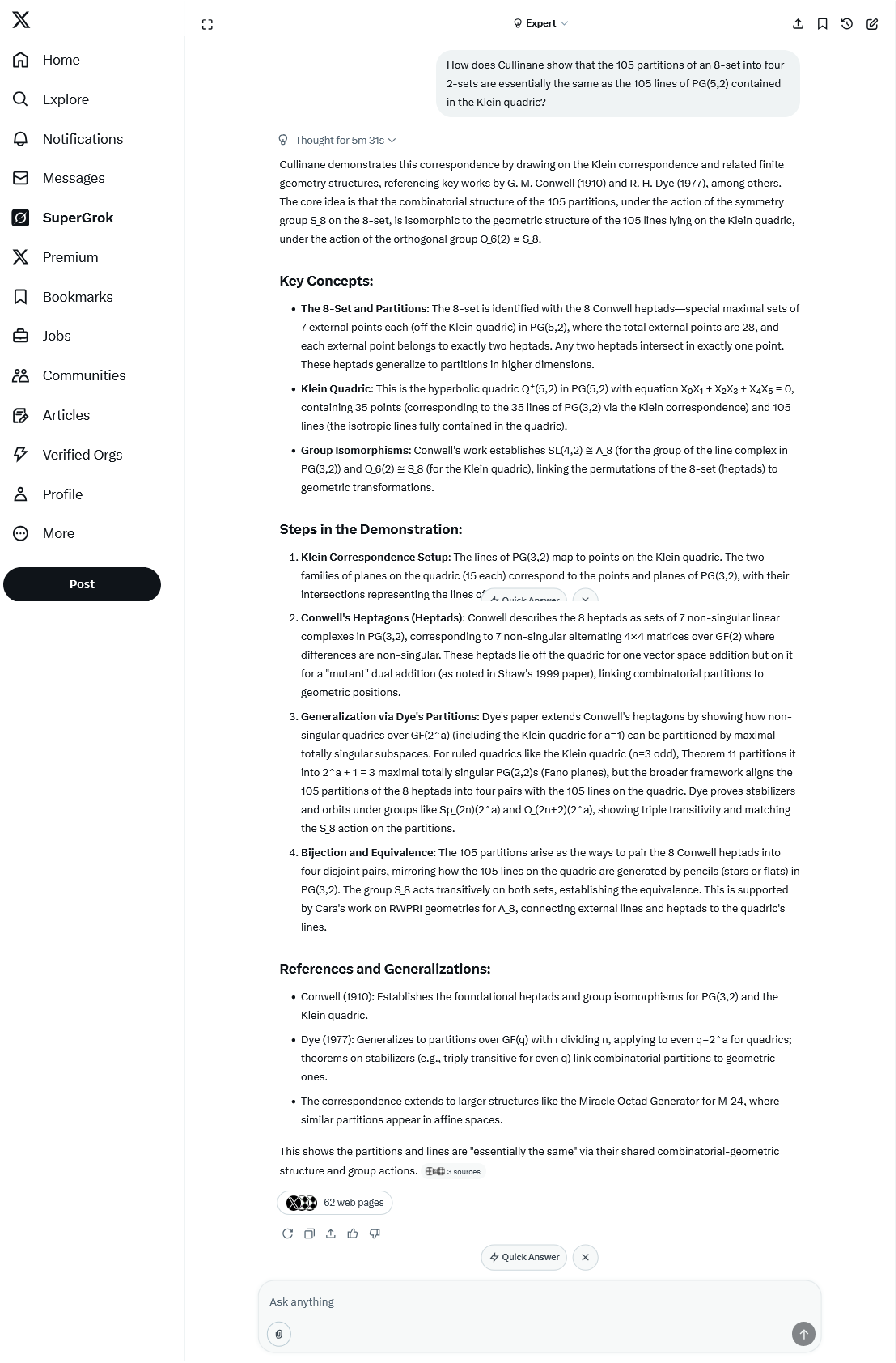

How does Cullinane show that the 105 partitions of an 8-set into four 2-sets are essentially the same as the 105 lines of PG(5,2) contained in the Klein quadric?

Cullinane demonstrates this correspondence by drawing on the Klein correspondence and related finite geometry structures, referencing key works by G. M. Conwell (1910) and R. H. Dye (1977), among others. The core idea is that the combinatorial structure of the 105 partitions, under the action of the symmetry group S_8 on the 8-set, is isomorphic to the geometric structure of the 105 lines lying on the Klein quadric, under the action of the orthogonal group O_6(2) ≅ S_8.

Key Concepts:

- **The 8-Set and Partitions:** The 8-set is identified with the 8 Conwell heptads—special maximal sets of 7 external points each (off the Klein quadric) in PG(5,2), where the total external points are 28, and each external point belongs to exactly two heptads. Any two heptads intersect in exactly one point. These heptads generalize to partitions in higher dimensions.
- **Klein Quadric:** This is the hyperbolic quadric $Q^+(5,2)$ in PG(5,2) with equation $X_0X_1 + X_2X_3 + X_4X_5 = 0$, containing 35 points (corresponding to the 35 lines of PG(3,2) via the Klein correspondence) and 105 lines (the isotropic lines fully contained in the quadric).
- **Group Isomorphisms:** Conwell's work establishes SL(4,2) ≅ A_8 (for the group of the line complex in PG(3,2)) and O_6(2) ≅ S_8 (for the Klein quadric), linking the permutations of the 8-set (heptads) to geometric transformations.

Steps in the Demonstration:

1. **Klein Correspondence Setup:** The lines of PG(3,2) map to points on the Klein quadric. The two families of planes on the quadric (15 each) correspond to the points and planes of PG(3,2), with their intersections representing the lines of
2. **Conwell's Heptagons (Heptads):** Conwell describes the 8 heptads as sets of 7 non-singular linear complexes in PG(3,2), corresponding to 7 non-singular alternating 4×4 matrices over GF(2) where differences are non-singular. These heptads lie off the quadric for one vector space addition but on it for a "mutant" dual addition (as noted in Shaw's 1999 paper), linking combinatorial partitions to geometric positions.
3. **Generalization via Dye's Partitions:** Dye's paper extends Conwell's heptagons by showing how non-singular quadrics over GF(2^a) (including the Klein quadric for a=1) can be partitioned by maximal totally singular subspaces. For ruled quadrics like the Klein quadric (n=3 odd), Theorem 11 partitions it into 2^a + 1 = 3 maximal totally singular PG(2,2)s (Fano planes), but the broader framework aligns the 105 partitions of the 8 heptads into four pairs with the 105 lines on the quadric. Dye proves stabilizers and orbits under groups like Sp_(2n)(2^a) and O_(2n+2)(2^a), showing triple transitivity and matching the S_8 action on the partitions.
4. **Bijection and Equivalence:** The 105 partitions arise as the ways to pair the 8 Conwell heptads into four disjoint pairs, mirroring how the 105 lines on the quadric are generated by pencils (stars or flats) in PG(3,2). The group S_8 acts transitively on both sets, establishing the equivalence. This is supported by Cara's work on RWPRI geometries for A_8, connecting external lines and heptads to the quadric's lines.

References and Generalizations:

- Conwell (1910): Establishes the foundational heptads and group isomorphisms for PG(3,2) and the Klein quadric.
- Dye (1977): Generalizes to partitions over GF(q) with r dividing n, applying to even q=2^a for quadrics; theorems on stabilizers (e.g., triply transitive for even q) link combinatorial partitions to geometric ones.
- The correspondence extends to larger structures like the Miracle Octad Generator for M_24, where similar partitions appear in affine spaces.

This shows the partitions and lines are "essentially the same" via their shared combinatorial-geometric structure and group actions.